# Random-Mass Dirac Fermions in an Imaginary Vector Potential: Delocalization Transition and Localization Length

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One dimensional system of Dirac fermions with a random-varying mass is studied by the transfer-matrix methods which we developed recently. We investigate the effects of nonlocal correlation of the spatial-varying Dirac mass on the delocalization transition. Especially we numerically calculate both the "typical" and "mean" localization lengths as a function of energy and the correlation length of the random mass. To this end we introduce an imaginary vector potential as suggested by Hatano and Nelson and solve the eigenvalue problem. Numerical calculations are in good agreement with the results of the analytical calculations.

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One of the most important problems in condensed matter physics is the localization phenomena in random-disordered systems [1]. At present it is believed that all states tend to localize in disorder systems in two and lower dimensions. In some special cases, however, some specific states remain extended even in the presence of strong disorders. System of Dirac fermions with a random-varying mass in one dimension has been studied from this point of view [2–8]. In the previous papers [9–11] we studied the effect of nonlocal correlation of the random mass on the extended states which exist near the band center. For numerical studies, we reformulate the system by transfer-matrix formalism, and obtained eigenvalues and wave functions for various configurations of random telegraphic mass [9]. We verified that the density of states obtained by the transfer-matrix methods is in good agreement with the analytical calculation in Ref. [10]. In this paper we shall introduce an imaginary vector potential into the system of the random-mass Dirac fermions and study a localization-delocalization phase transition by varying the magnitude of the vector potential. Through this study we shall obtain the localization length of the states as a function of energy and the correlation length of the random mass. This method of calculating the localization length is based on the idea by Hatano and Nelson [12].

We shall consider a Dirac fermion in one spatial dimension with a coordinate-dependent mass $m(x)$ and in an imaginary vector potential $g$, whose Hamiltonian is given by,

$$\mathcal{H} = \int dx \psi^\dagger h \psi, \tag{1}$$

$$h = -i\sigma^z(\partial_x + g) + m(x)\sigma^y, \tag{2}$$

where $\vec{\sigma}$ are the Pauli matrices. This fermion model is an low-energy effective model of random-hopping tight binding models and random-bond spin chains [5]. We introduce the components of $\psi$ as $\psi = (u, v)$. In terms of them the Dirac equation is given as,

$$\begin{aligned} \Big( \frac{d}{dx} + g + m(x) \Big) u(x) &= Ev(x), \\ \Big( -\frac{d}{dx} - g + m(x) \Big) v(x) &= Eu(x). \end{aligned} \tag{3}$$

(We follow the notations in Ref. [3].) From Eqs.(3), we obtain the Schrödinger equations,

$$\Big( -\frac{d^2}{dx^2} - 2g\frac{d}{dx} - m'(x) + (m^2(x) - g^2) \Big) u(x) = E^2 u(x), \tag{4}$$

and similarly for $v(x)$.

In this paper we restrict the shapes of $m(x)$ to multi-soliton-antisoliton configurations [13]. The multi-soliton-antisoliton configurations are given by,

$$\begin{aligned} m(x) = &\sum_i \bar{m}(\theta(x - \alpha_i) - 1) \\ &+ \sum_j \bar{m}(\theta(-x + \beta_j) - 1), \end{aligned} \tag{5}$$

where $\alpha_i$'s($\beta_j$'s) are positions of solitons(anti-solitons). An example of $m(x)$ is given in Fig.1. If we vary the distances $l$ between soliton and anti-soliton according to the exponential distribution like,

$$P(l) = \frac{1}{2\tilde{\lambda}} \exp\Big( -\frac{l}{2\tilde{\lambda}} \Big), \tag{6}$$

where $\tilde{\lambda}$ is a parameter, then $m(x)$ has the following correlation [3],

$$[\ m(x)\ m(y)\ ]_{\rm ens} = \frac{A}{\tilde{\lambda}} \exp\ (-|x - y|/\tilde{\lambda}), \tag{7}$$

where $\sqrt{\frac{A}{\tilde{\lambda}}}$ essentially corresponds to the height of the soliton and anti-soliton, i.e., $\bar{m}$ in (5). From (7), $\tilde{\lambda}$ is

the correlation length of the random mass and the limit $\tilde{\lambda} \to 0$ corresponds to the white-noise case. In subsequent papers, we shall study Dirac fermions with long-range correlated random mass by using the methods examined in this paper [14]. There we expect some interesting phenomena like existence of nontrivial mobility edge, nonuniversality of the multi-fractal exponents, etc. Studies in this paper show that the imaginary-vector-potential methods for calculating the localization lengths are reliable and we shall use them for studies on random systems with long-range correlated disorder.

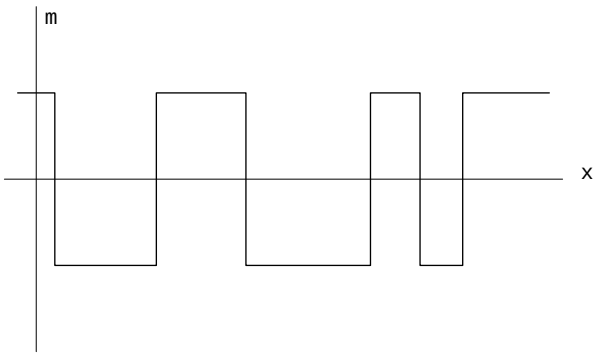


FIG. 1. An example of configurations of solitons and anti-solitons.

For the vanishing imaginary vector potential, we solved the Schrödinger equations in (4) under the periodic boundary condition with various multi-soliton-antisoliton configurations of $m(x)$ using the transfer-matrix method, and obtained the energy spectrum and wave functions [9].

Effect of the imaginary vector potential was discussed by Hatano and Nelson [12]. Let us denote the eigenfunction of energy $E$ for $g = 0$ as $\Psi_0(x)$, and suppose the shape of $\Psi_0(x)$ as

$$\Psi_0(x) \cong \exp\left(-\frac{|x - x_c|}{\xi_0}\right), \tag{8}$$

where $\xi_0$ is the localization length and $x_c$ is the center of this localized state. When we turn on the constant imaginary vector potential $g$, the eigenfunction is obtained from $\Psi_0(x)$ by the "imaginary" gauge transformation,

$$\Psi(x) \cong \exp\left(-\frac{|x - x_c|}{\xi_0} - \ g\ (x - x_c)\right). \tag{9}$$

This means that the localization length of this eigenfunction is

$$\xi_g = \frac{\xi_0}{1 + g\xi_0}\ (x > x_c)$$
$$\xi_g = \frac{\xi_0}{1 - g\xi_0}\ (x < x_c). \tag{10}$$

At the point $g = 1/\xi_0$, localization length for $x < x_c$ diverges, and if the imaginary vector potential $g$ is increased more, $\xi_g$ for $x < x_c$ becomes negative, and this eigenfunction cannot satisfy the periodic boundary condition even if the length of system is large enough. So if localized eigenstate with energy $E$ disappears at $g = g_c$ as $g$ is increased, then the localization length of the eigenstate $\Psi_0(x)$ is $1/g_c$. Actually it is shown that for $g > g_c$, energy eigenvalue of the state has an imaginary part and the state is extended as we shall see shortly [12].

The transfer-matrix methods can be easily extended for the case of nonvanishing $g$. We obtain energy eigenvalues and eigenfunctions numerically. First of all, we show that the delocalization transition actually occurs. In Fig.2, we show the wave functions of a low-lying state in vanishing and nonvanishing imaginary vector potential. For $g = 0$, the state is obviously localized whereas at $g = 0.03$ the state becomes extended and the energy eigenvalue has an imaginary part. As discussed in Ref. [12], the density distribution of a particle is given by $|\Psi(x, -g)\Psi(x, g)|$ where $\Psi(x, -g)$ is equal to the left eigenfunction.

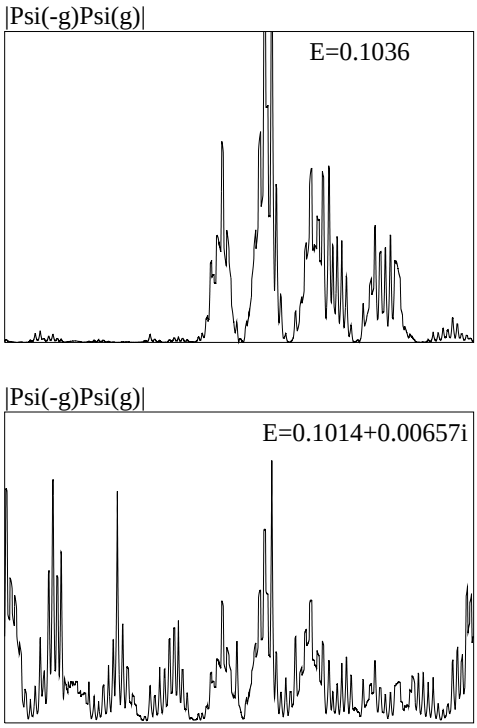


FIG. 2. An example of the localized and the extended wave functions in vanishing and nonvanishing imaginary vector potential $g = 0$ and $g = 0.03$. The state of $g = 0.03$ has a complex energy eigenvalue.

Let us turn to the localization length. For the white-noise case $[m(x)m(y)]_{ens} = A\ \delta(x - y)$, "typical" local-

ization length or the inverse of the Lyapunov exponent was obtained as [3,15]

$$\xi_t(E) = |\ln E/2A|. \tag{11}$$

Numerically the typical localization length $\xi_t(E)$ is obtained by averaging over localization lengths of *all* eigenstates with energy $E$. On the other hand, Balents and Fisher calculated the averaged Green function and obtained the mean localization length from the spatial decay of the Green function [5]. The result is

$$\xi_m(E) = |\ln E/2A|^2. \tag{12}$$

Case of nonlocally correlated random mass was studied in Refs. [10,11] and $\xi_m(E)$ is obtained as a function of $\tilde{\lambda}$ in Eq.(7).

By numerical calculation we obtain both the typical and mean localization lengths. As we mentioned above, the typical localization length is the average over all solutions of the Schrödinger equation (4), whereas the mean localization length is determined by the states which make dominant contributions to the Green function, i.e., which have large localization length.

The result of the numerical calculation of the typical localization length of the white-noise case is given in Fig.3. We show the ratio of the numerical results to the analytical calculation in Eq.(11) in order to compare these two results. Therefore if the energy dependences of the localization lengths obtained numerically and analytically are the same, this ratio should be constant. In Fig.3, the ratio seems constant over the whole range of $E$.

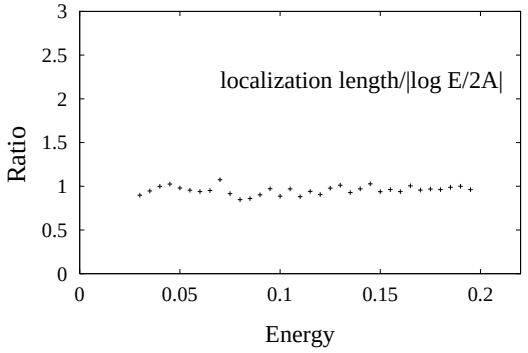


FIG. 3. The comparison between the analytical and numerical results of the typical localization length for the (almost) white-noise case, i.e., small-$\tilde{\lambda}$ case. In the numerical calculation, we set $L$(system size)= 50, $\tilde{\lambda} = 1/60$, and the energy slice $\delta E = 0.02$. This result is averaged over 2500 trials. The analytical and numerical results are in good agreement.

In Fig.4, we show the numerical results of the "mean" localization length. Here we use the solutions to the Schrödinger equation which have long localization length. More precisely, "large" localization length $\xi$ means the one which satisfies $\xi >$ (the "typical" localization length)+ 1.5 $\sigma$ in each energy slice. From Fig.4, we can conclude that the energy dependence of the mean localization length obtained numerically is in agreement with Eq.(12).

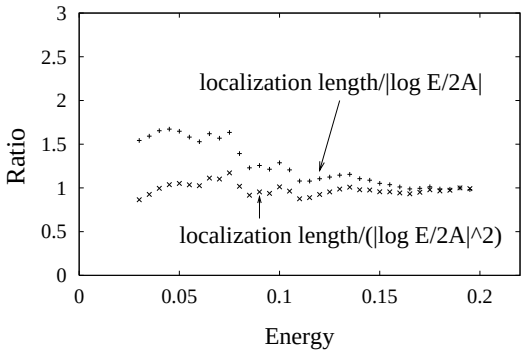


FIG. 4. The comparison between the analytical and numerical results of the mean localization length. We use the data of the states which have large localization length. (See the text.) We show the ratios of the numerical calculations both to Eqs.(11) and (12). The ratios are normalized at $E = 0.19$.

From the above studies, we can conclude that the above methods of calculating the localization lengths are reliable.

We shall turn to the case of the nonlocally-correlated disorder. In the white-noise limit ($\tilde{\lambda} = 0$ case in Eq.(7)), the localization lengths diverge only at $E = 0$, that is, extended states exist only at $E = 0$. If we let $\tilde{\lambda} > 0$, the random mass becomes nonlocally-correlated, and the critical energy or the mobility edge at which the delocalization transition occurs may change.

We investigate the "typical" and "mean" localization lengths in the case of nonvanishing $\tilde{\lambda}$'s. The behaviour of the "typical" and "mean" localization lengths obtained as in the white-noise case are given in Figs.5 and 6. It seems that there is *no* $\tilde{\lambda}$-dependence in the typical localization length. On the other hand, Fig.6 shows that the mean localization length has a small but finite dependence on $\tilde{\lambda}$.

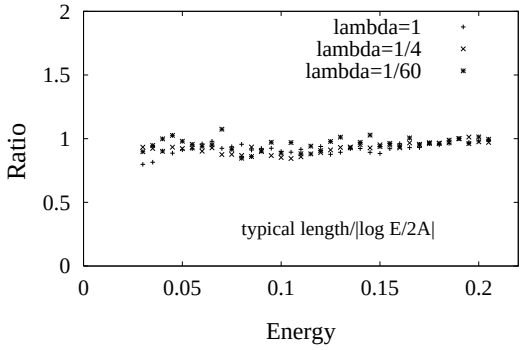

FIG. 5. The behaviour of the localization length in the case of nonvanishing $\tilde{\lambda}$'s. The ratio is the "typical" localization length to $|\ln E|$. The ratio is normalized at $E = 0.19$.

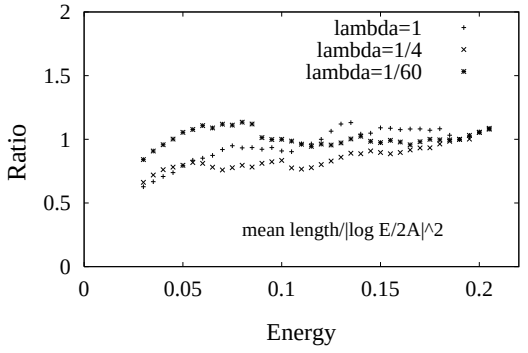


FIG. 6. The behaviour of the localization length in the case of nonvanishing $\tilde{\lambda}$'s. The ratio is the "mean" localization length to $|\ln E|^2$. The energy slice $\delta E = 0.03$, and the ratio is normalized at $E = 0.19$.

From the above calculations, we conclude that the effect of the short-range correlations in disorders is not so large. Especially the result indicates that the delocalization transition occurs at $E < 0.03$. (If the mobility edge exists at $E_c > 0$, the ratio in Fig.5 or 6 must diverge at $E_c$.) The delocalization transition probably occurs at $E = 0$.

In the previous paper [11] we calculated the localization length for the random mass with the short-range correlation (7). We obtained the "mean" localization length to the 1st order of $\tilde{\lambda}$ by means of the Green function method. The result is

$$\xi(E) = \frac{1}{A}\Big(\frac{\ln|\frac{E}{2A}|^2}{\pi^2} + A\tilde{\lambda}\frac{4|\ln\frac{E}{2A}|}{\pi^2}\Big) + O(\tilde{\lambda}^2). \qquad (13)$$

In Fig.7 we show the ratios of the numerical result to the analytical calculation up to the 0th and the 1st order of $\tilde{\lambda}$. This shows that the analytical result with the 1st order correction of $\tilde{\lambda}$ is in better agreement with the numerical result, but the correction is small.

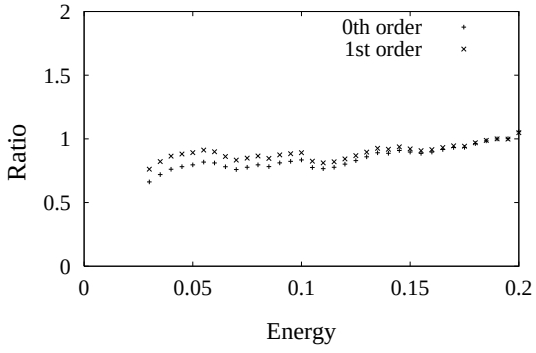


FIG. 7. The comparison between the numerical result and the analytical one in the case of relatively large $\tilde{\lambda}$. We show the ratio of the localization length calculated numerically to the ones obtained analytically. Here we used the analytical results of the 0th and the 1st order of $\tilde{\lambda}$. Here we set $\tilde{\lambda} = 1/4$ and $L = 50$. The energy slice $\delta E = 0.03$, and the ratio is normalized at $E = 0.19$.

From the investigations given as far we can conclude that the numerical methods used in this paper are reliable for calculating the localization lengths. It is very interesting to study the case of disorders with a long-range correlation. We expect that a nontrivial mobility edge $E_c > 0$ exists for a certain long-range correlated random mass. Actually the one-dimensional Anderson model with long-range correlated disorder was studied [16], and it is shown that there exists a nontrivial mobility edge.

We can also calculate exponents of the multi-fractal scaling [17] by the transfer-matrix methods [9]. These problems are under study and results will be reported in a future publication [14].

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